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High-temperature series for classical n-vector models with general anisotropy

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Abstract. A compact parametrization is derived for the high-temperature series expansions of the free energy and two-spin correlation function of classical n-vector models with completely general anisotropic pair interactions. The number of independent coefficients needed for general anisotropy is much less than required in conventional expansions. The series for the free energy of the Heisenberg model ($n = 3$) on three cubic lattices is presented to tenth order as an example.

1. Introduction and summary

In the past high-temperature series have been derived for a large number of lattice spin models‡. In this note we use a method for calculating graphical weights (Gerber and Fisher 1975) to obtain a compact parametrization for many of these series. The Hamiltonian which applies to a large class of classical n-vector models has the reduced form

$$-\beta\mathscr{H} = K \sum_{(i,j)} \sum_{\alpha,\beta=1}^{n} s_i^\alpha U_{\alpha\beta} s_j^\beta - \sum_i w(|\boldsymbol{s}_i|^2) \tag{1.1}$$

where $\beta = (k_B T)^{-1}$. The $\boldsymbol{s}_i = (s_i^1, \ldots, s_i^n)$ are n-component classical spins located on the sites of a lattice. The quantity $\hat{U}$ is an $n \times n$ matrix which specifies the anisotropic nature of the bilinear coupling between the pairs (i, j) of neighbouring spins. The second term in (1.1) allows for an arbitrary spin length weighting factor of the form $\exp[-w(|\boldsymbol{s}|^2)]$ for each spin. This Hamiltonian includes many of the previously treated spin models such as the Ising model ($n = 1$), the fixed length plane rotator and X–Y models (see, eg, Ferrer *et al* 1973). Since we may consider the Ising model for arbitrary spin S as an $n = 1$ model with a spin-weight factor composed of a series of delta functions, the Hamiltonian (1.1) also describes these models. The models for which our parametrization is most useful are anisotropic models, eg the Heisenberg ($n = 3$) model, which for some cases of anisotropy have been analysed by Pfeuty *et al* (1974) and which have the general Hamiltonian

$$-\beta\mathscr{H} = K \sum_{(i,j)} (g_x s_i^x s_j^x + g_y s_i^y s_j^y + g_z s_i^z s_j^z) \tag{1.2}$$

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‡ For an extensive review of high-temperature expansions see the volume edited by Domb and Green (1974).

where fixed length spins $|s| = 1$ have been assumed. The high-temperature expansion for the free energy may be written

$$-\beta F(K) = \sum_{l=0}^{\infty} b_l K^l. \tag{1.3}$$

The lth-order coefficient b_l is a homogeneous polynomial of order l in the anisotropy parameters g_i in (1.2). In order to write down this polynomial one is required to specify its roughly $\frac{1}{2}l^2$, apparently independent, coefficients for the most general case. The method for calculating the series (1.3), which we derive in § 2, expresses the coefficients b_l only in terms of the power traces or moments

$$u_k \equiv \mathrm{Tr}(\hat{U}^k) \qquad \text{for } 2 \leqslant k \leqslant l. \tag{1.4}$$

The possibility of such a result might well have been expected on symmetry grounds. For the anisotropic Heisenberg model (1.2) the u_k are simply given by the power sums

$$u_k = g_x^k + g_y^k + g_z^k. \tag{1.5}$$

The coefficient b_l is found to be a linear combination of products of the form

$$v(l, \{k_i\}) \equiv u_{k_1} u_{k_2} \ldots, \qquad \text{with } \sum_i k_i = l. \tag{1.6}$$

The set $\{k_i\}$ consists of positive integers k_i. A particular value of k_i can, of course, appear several times in the set $\{k_i\}$. The linear combination of $v(l, \{k_i\})$ which determines b_l has the form

$$b_l = \sum_{\{k_i\}, \Sigma k_i = l} B_l\{k_i\} v(l, \{k_i\}). \tag{1.7}$$

The labels of the $B_l\{k_i\}$ listed in the tables of the appendix provide examples of allowed sets $\{k_i\}$. Because of the condition $\Sigma k_i = l$ the number of coefficients $B_l\{k_i\}$ in a given order l is considerably smaller than the roughly $\frac{1}{2}l^2$ coefficients needed when b_l is represented as a homogeneous polynomial in the g_i. The number of $B_l\{k_i\}$ can be further reduced for certain lattices since only those values of k_i are allowed for which a closed k_i-step polygon occurs on a lattice. Hence generally $k_i = 1$ never occurs, and for loose-packed lattices the k_i must be all even. As an example we mention that the coefficient b_{10} requires about fifty coefficients in the conventional polynomial representation, whereas the number of non-vanishing coefficients $B_{10}\{k_i\}$ in (1.7) is twelve for close-packed lattices (with triangles, pentagons, etc) but only seven for loose-packed lattices. The number of coefficients $B_l\{k_i\}$ is furthermore independent of the number of components n, whereas their values depend on n as well as on features of the spin-weight factor.

The spin-weight factor $\exp[-w(|s|^2)]$ in equation (1.1) enters the expansion to lth order only through the reduced spin moments of the form

$$f_m(n) = \frac{\langle |s|^{2m} \rangle_0 Q(2m, n)}{(\langle |s|^2 \rangle_0 Q(2, n))^m} \qquad \text{with } 2m \leqslant l \tag{1.8}$$

where the non-interacting ($T = \infty$) spin moments are defined by

$$\langle |s|^p \rangle_0 = \frac{\int_0^\infty s^{p+n-1} \mathrm{e}^{-w(s^2)} \mathrm{d}s}{\int_0^\infty s^{n-1} \mathrm{e}^{-w(s^2)} \mathrm{d}s} \tag{1.9}$$

and the simple combinatorial factors are

$$Q(2m, n) = \Gamma(\tfrac{1}{2}n)/\Gamma(m+\tfrac{1}{2}n). \tag{1.10}$$

In practical expansions it is convenient to normalize the interaction strength (see Gerber and Fisher 1975) by setting

$$\bar{K} = K\langle|s|^2\rangle_0/n. \tag{1.11}$$

In terms of the anisotropy traces u_k and spin moments $f_m(n)$ defined above, we may write an expansion for the free energy on a general lattice by introducing the lattice constants p_i, for closed graphs of i bonds (see appendix III, table A of Domb 1960) and $p_{i\cdot j}$ for open graphs of i bonds (where $i \cdot j$ corresponds to the labels $(i \cdot j)$ in table B of appendix III of Domb 1960). We find the result to be

$$\begin{aligned}-\beta F(K) = {} & \tfrac{1}{2}p_{1\cdot 1}u_2\bar{K}^2 + p_3u_3\bar{K}^3 + \bar{K}^4\{u_4(p_4+\tfrac{1}{4}p_{1\cdot 1}f_2(n)^2+\tfrac{1}{2}p_{2\cdot 1}f_2(n)) \\ & + u_2^2[\tfrac{1}{8}p_{1\cdot 1}(f_2(n)^2-1)+\tfrac{1}{4}p_{2\cdot 1}(f_2(n)-1)]\} + \bar{K}^5\{u_5(p_5+3p_3f_2(n)^2 \\ & + p_{4\cdot 1}f_2(n)) + u_3u_2[\tfrac{3}{2}p_3(f_2(n)^2-1)+\tfrac{1}{2}p_{4\cdot 1}(f_2(n)-1)]\} + O(\bar{K}^6).\end{aligned} \tag{1.12}$$

The derivation of this result utilizes the recursion relation method introduced by Stanley and Kaplan (1966) (Stanley 1974); it is presented in § 2. The explicit series in terms of the variable K for the Heisenberg model with $\langle|s|^2\rangle_0 = 1$ on the three cubic Bravais lattices are tabulated in the appendix to tenth order in K.

The high-temperature expansion of the two-spin correlation function between spins at sites i and j may be expressed as

$$\Gamma_{ij}(\hat{R}) \equiv \sum_{\alpha,\beta}\langle s_i^\alpha R_{\alpha\beta}s_j^\beta\rangle = \sum_{l=0}^{\infty} a_{ij,l}(\hat{R})K^l \tag{1.13}$$

where the matrix $\hat{R}$ specifies the particular combination of correlations $\langle s_i^\alpha s_j^\beta\rangle$ of interest. The corresponding expansion coefficient $a_{ij,l}(\hat{R})$ can be written in equally transparent form. The only new quantities which enter are the modified traces

$$r_p = \mathrm{Tr}(\hat{R}\hat{U}^p). \tag{1.14}$$

The index p runs from a minimal value $p_{\min}$ equal to the smallest number of steps by which one can reach site j from site i, to a maximal value $p_{\max} = l$ (one may have $p_{\min} = 0$). The coefficient $a_{ij,l}(\hat{R})$ in (1.13) is again a linear combination of products similar to (1.6), namely

$$a_{ij,l}(\hat{R}) = \sum_{p,\{k_q\}} A_{ij,l}(p, \{k_q\}, \hat{R})r_p\prod_q u_{k_q}. \tag{1.15}$$

The index set $\{k_q\}$ now obeys the expected relation

$$p + \sum_q k_q = l \tag{1.16}$$

where, as before, the same positive number k can appear several times in $\{k_q\}$. This condition likewise reduces the number of coefficients $A_{ij,l}(p, \{k_q\}, \hat{R})$ to a conveniently small value. Using the same notation as in (1.12) the reduced susceptibility

$$\chi_0(\hat{R}) \equiv k_B T^2\chi(\hat{R})/m^2$$

may be written

$$\chi_0(\hat{R})n/\langle|s|^2\rangle_0 = r_0 + 2r_1p_{1\cdot1}\bar{K} + \bar{K}^2[r_2(2p_{2\cdot1} + 2p_{1\cdot1}f_2(n)) + r_0u_2p_{1\cdot1}(f_2(n)-1)]$$
$$+ \bar{K}^3\{r_3(2p_{3\cdot2} + 6p_3f_2(n) + 4p_{2\cdot1}f_2(n) + 2p_{1\cdot1}f_2(n)^2)$$
$$+ r_1u_2[2p_{2\cdot1}(f_2(n)-1) + p_{1\cdot1}(f_2(n)^2-1)]$$
$$+ r_0u_3 3p_3[f_2(n)-1]\} + O(\bar{K}^4). \tag{1.17}$$

The main steps of the derivation of this form are presented in § 3. We have not, so far, derived the correlation series to the same high order as the free energy series.

It is clear that from expansions like (1.12) and (1.17), one can immediately obtain series for any quantity which is derived from the free energy or correlation function by taking derivatives with respect to K or any parameter occurring in the matrix $\hat{U}$. This includes in particular the specific heat, anisotropy susceptibilities (Pfeuty *et al* 1974) etc. In addition, by using the explicit correspondence between the limit $n \to 0$ and the problem of interacting random walks (Gerber and Fisher 1975) and setting $r_i = 1$ and $u_i = 0$, we can obtain from (1.13) and (1.17) various generating functions for interacting random walks. The weight factors f_m defined in equation (1.8) must then be reinterpreted as the Boltzmann factors for an $(m-1)$-fold self-intersection of a walk at a given vertex as explained in Gerber and Fisher (1975) (see also Jasnow and Fisher 1975).

2. Free energy expansion

Starting with the Hamiltonian (1.1) we follow the procedure developed by Stanley and Kaplan (1966) (Stanley 1974) and construct a high-temperature expansion for the free energy in the form

$$-F(K)/k_BT = \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}(e^{-\beta\mathscr{H}}) = \sum_{l=0}^{\infty} b_lK^l. \tag{2.1}$$

The coefficients b_l may be written as a sum of contributions of the form

$$b_l(\hat{U}, \mathscr{L}) = \sum_{G_l} (G_l, \mathscr{L})\tilde{B}(\hat{U}, G_l) \tag{2.2}$$

where the sum runs over multigraphs G_l (Essam and Fisher 1970) with l bonds (counting multibonds with appropriate multiplicity) which (i) are connected and (ii) have only vertices of even degree (Stanley 1974). The quantity $(G, \mathscr{L})$ denotes the weak lattice constant per site (Essam and Fisher 1970) of the graph G on the lattice $\mathscr{L}$. The graphical weights $\tilde{B}$ obey recursion relations in terms of proper subgraphs G_k of G_l which fulfil the same conditions as G_l. If G_{l-k} denotes the complementary subgraph of $l-k$ lines the recursion relations are

$$\tilde{B}(\hat{U}, G_l) = M(\hat{U}, G_l) - \sum_{G_k} \frac{k}{l} \tilde{B}(\hat{U}, G_k)M(\hat{U}, G_{l-k}) \tag{2.3}$$

as shown by Stanley and Kaplan (1966) (Stanley 1974). The graphical weights M which enter here are defined through the averages

$$M(\hat{U}, G) = \left\langle \prod_{(i,j)} \left(\sum_{\alpha,\beta} s_i^\alpha U_{\alpha\beta} s_j^\beta \right)^{m_{ij}} (m_{ij}!)^{-1} \right\rangle_0 \tag{2.4}$$

defined in analogy to (1.9). In this expression the product runs over all pairs of vertices, labelled by i and j, of the graph G and the multiplicity of the bond joining the two vertices i and j is denoted by m_{ij}.

In order to calculate these M weights or equivalently the P weights,

$$P(\hat{U}, G) = \left\langle \prod_{(i,j)} \left(\sum_{\alpha,\beta} s_i^\alpha U_{\alpha\beta} s_j^\beta \right)^{m_{ij}} \right\rangle_0 , \tag{2.5}$$

we utilize a reduction formula for vertex weights which was derived recently (Gerber and Fisher 1975). The basic integral required for a single selected spin s, 'coupled' to $2m$ adjacent spins s_j is

$$\mathscr{I}(\hat{U}) = \int_0^\infty \frac{\mathrm{d}s}{I(n)} s^{n-1} \mathrm{e}^{-w(s^2)} \int \frac{\mathrm{d}\Omega(s)}{\omega_n} \prod_{j=1}^{2m} \left(\sum_{\alpha,\beta} s^\alpha U_{\alpha\beta} s_j^\beta \right) \tag{2.6}$$

where

$$I(n) = \int_0^\infty s^{n-1} \mathrm{e}^{-w(s^2)} \mathrm{d}s \tag{2.7}$$

and where $\mathrm{d}\Omega(s)$ denotes the surface element of an n-dimensional unit sphere of surface ω_n. This integral may be reduced to a sum over all distinct pairings of the $2m$ spins s_j coupled to s, into m pairs with indices $[p(2j-1), p(2j)]$ with $j = 1, \ldots, m$. There are a total of $(2m)!/2^m m!$ such distinct pairings. The reduction formula proved in Gerber and Fisher (1975, see explicitly equations (29)–(33) and the appendix) is then

$$\mathscr{I}(\hat{U}) = f_m(n) \sum_{\text{pair}} \prod_{j=1}^{m} \left[\int_0^\infty \frac{\mathrm{d}s}{I(n)} s^{n-1} \mathrm{e}^{-w(s^2)} \int \frac{\mathrm{d}\Omega(s)}{\omega_n} \left(\sum_{\alpha,\beta} s^\alpha U_{\alpha\beta} s_{p(2j-1)}^\beta \right) \left(\sum_{\gamma,\delta} s^\gamma U_{\gamma\delta} s_{p(2j)}^\delta \right) \right]. \tag{2.8}$$

We now apply this reduction formula to each vertex of valence 4 or more in a graph G_l. Each distinct set of choices of pairings then yields a decomposition of the graph G into a set of closed, effectively non-intersecting polygons (see figure 1). If the number of bonds in the ith polygon of this set is k_i we may describe the set by the set $\{k_i\}$. The relation

$$\sum k_i = l \tag{2.9}$$

is, of course, satisfied since each of the l bonds of G_l appears in one and only one polygon of the decomposition. The reduction formula then states that the contribution of a particular selection of pairings (ie a particular polygonal decomposition) is simply equal to the product of P weights of the constituent polygons. In addition each vertex i of G_l contributes a factor $f_{m_i}(n)$ which accounts for its multiplicity.

The P weight of a k-step polygon P_k is readily found to be

$$P(\hat{U}, P_k) = (\langle |s|^2 \rangle_0 / n)^k u_k \tag{2.10}$$

where the basic power traces u_k are as defined in (1.4) (see equation (30) of Gerber and Fisher 1975). The particular polygonal decomposition under consideration then gives a contribution

$$(\langle |s|^2 \rangle_0 / n)^l \prod_i u_{k_i} \tag{2.11}$$

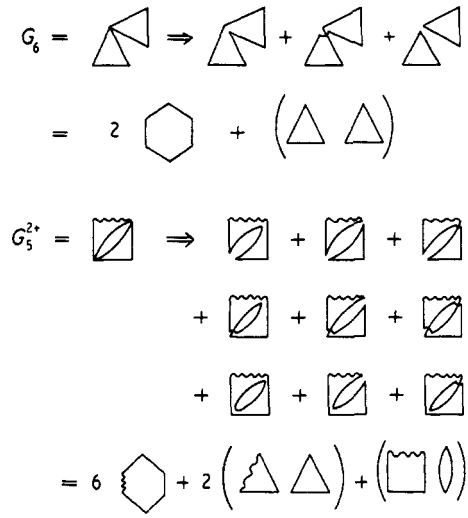

Figure 1. Polygonal decompositions of a free energy graph G_6 with one vertex of multiplicity 4 and of a correlation function graph G_5^2 with two vertices of multiplicity 4. One obtains the decomposition numbers $c(G_6, 1_6) = 2$ and $c(G_6, 2_3) = 1$ (see (2.12)); and $c(G_5^{2+}, 5) = 6$, $c(G_5^{2+}, 3, 1_2) = 1$ and $c(G_5^{2+}, 2, 1_3) = 2$ (see (3.4)).

to the total weight. Summing over all polygonal decompositions and taking into account the vertex-weighting factors $f_m(n)$ yields the final expression

$$P(\hat{U}, G_l) = \left(\frac{\langle |s|^2 \rangle_0}{n}\right)^l \prod_i f_{m_i}(n) \sum_{\{n_k\}} c(G_l, \{n_k\}) \prod_k u_k^{n_k} \tag{2.12}$$

where the basic *decomposition numbers* $c(G_l, \{n_k\})$ of the graph G_l represent the number of times the polygonal decomposition procedure leads to exactly n_2 two-step polygons, n_3 three-step polygons, and so on. From equation (2.11) we have a necessary condition for a non-vanishing $c(G_l, \{n_k\})$, namely

$$\sum_k k n_k = l. \tag{2.13}$$

This is identical with the original condition (2.9). The evaluation of the $\tilde{B}$ weights in equation (2.2) by using the recursion relations (2.3) is a straightforward but tedious business. It is clear from (2.3) and (2.12), however, that the only parameters in the high-temperature expansion of the free energy in lth order which refer to a particular matrix $\hat{U}$ are the products (1.6) or equivalently the products

$$\prod_k u_k^{n_k} \tag{2.14}$$

where the $\{n_k\}$ satisfy (2.13). This parametrization is the one exhibited in equation (1.12) and used in the appendix for the Heisenberg model.

3. Two-point correlation function

The procedures for calculating high-temperature expansions for the correlation functions (Stanley 1974) follow closely those for the free energy expansion. We consider the

correlation of two spins at sites i and j determined by a matrix $\hat{R}$ through (1.13). The expansion coefficients $a_{ij,l}(\hat{R})$ in (1.13) may again be written as a sum of the form

$$a_{ij,l}(\hat{R}) = \sum_{G_l^2} (G_l^2, \mathscr{L})^{ij} \tilde{A}(\hat{R}, \hat{U}, G_l^2). \tag{3.1}$$

The sum now runs over all the graphs G_l^2 which (i) are connected and (ii) have precisely two vertices of odd degree (as indicated by the superscript 2). The weak lattice constant $(G_l^2, \mathscr{L})^{ij}$ obeys the additional constraint that the two odd vertices of G_l^2 have to be located at the sites i and j of the lattice. The $\tilde{A}$ weights obey the recursion relations (Stanley and Kaplan 1966, Gerber and Fisher 1974)

$$\tilde{A}(\hat{R}, \hat{U}, G_l^2) = N(\hat{R}, \hat{U}, G_l^2) - \sum_{G_k^2} \tilde{A}(\hat{R}, \hat{U}, G_k^2) M(\hat{U}, G_{l-k}) \tag{3.2}$$

where again the sum runs over all proper subgraphs G_k^2 of G_l^2 which fulfil the same conditions as G_l^2 and which have complementary subgraphs G_{l-k} with only even vertices. The M weights are the same as defined in (2.4) but the N weights are defined by

$$N(\hat{R}, \hat{U}, G_l^2) = \left\langle \sum_{\alpha,\beta} s_0^\alpha R_{\alpha\beta} s_1^\beta \prod_{(i,j)} \left(\sum_{\gamma,\delta} s_i^\gamma U_{\gamma\delta} s_j^\delta \right)^{m_{ij}} (m_{ij}!)^{-1} \right\rangle_0 . \tag{3.3}$$

The subscripts 0 and 1 label the two odd vertices and the rest of the notation is as in (2.4). The method used to evaluate the P weights in (2.5) can similarly be applied to the N weights. It is convenient to add to the graphs G_l^2 a 'wavy' line joining the two odd vertices. This line represents the matrix $\hat{R}$ and augments the graph G_l^2 to one G_l^{2+} having only even vertices. If one applies the polygonal decomposition procedure described in § 2 to the graphs G_l^{2+} (see figure 1), one always obtains one polygon (of say $p+1$ steps) which contains the wavy line and hence gives an expectation value of $(\langle |s|^2 \rangle_0 / n)^{p+1} r_p$ (compare with (1.14)) and several polygons of only $\hat{U}$-type (straight) lines with expectation values given by (2.10). Summing over all the polygonal decompositions of G_l^{2+} as outlined in § 2 then gives the final expression

$$N(\hat{R}, \hat{U}, G_l^2) = (\langle |s|^2 \rangle_0 / n)^{l+1} \prod_i f_{m_i}(n) \prod_{(i,j)} (m_{ij}!)^{-1} \sum_{p,\{n_k\}} c(G_l^{2+}, p, \{n_k\}) r_p \prod_k u_k^{n_k}. \tag{3.4}$$

The basic *decomposition numbers* $c(G_l^{2+}, p, \{n_k\})$ for the graph G_l^{2+} now represent the number of times the polygonal decomposition procedure, applied to G_l^{2+}, leads to a $(p+1)$-step polygon containing the wavy line, and n_2 two-step polygons, n_3 three-step polygons, etc, with only $\hat{U}$-type lines. The range of allowable p values has been mentioned after equation (1.14) and the n_k fulfil the condition

$$p + \sum_k k n_k = l \tag{3.5}$$

which is equivalent to (1.16). The products over vertex pairs (i,j) in (3.4) take care of the combinatorial factors arising from multibonds of $\hat{U}$-type lines with multiplicity m_{ij}. For the product of vertex weights $f_m(n)$ the multiplicity of the originally odd vertices has to be increased by one due to the wavy line. The m thus satisfy the relation

$$\sum_i m_i = 2(l+1). \tag{3.6}$$

From (3.4) and (3.2) it is clear that the lth-order coefficient in the expansion (1.13) has, in fact, the compact parametrization in terms of the r_p and u_k explained in § 1 (see (1.15)). The expansion (1.17) illustrates the type of results obtainable by this method.

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Appendix. Expansion coefficients for the free energy

Coefficients $B_l\{k_i\}$, defined in (1.7) and (1.6) for the free energy of the classical Heisenberg model ($n = 3$, unit spins) on three cubic lattices (FCC: face-centred cubic; BCC: body-centred cubic; SC: simple cubic). The numbers in brackets are decimal exponents.

Table 1.

$B_l\{k_i\}$	FCC	$B_l\{k_i\}$	FCC
$B_2\{2\}$	$\frac{1}{3}$	$B_9\{9\}$	5·282115545 (1)
$B_3\{3\}$	$\frac{8}{27}$	$B_9\{7, 2\}$	−1·240611433 (1)
$B_4\{4\}$	6·585185$\overline{185}$ (−1)	$B_9\{6, 3\}$	−5·528904338 (0)
$B_4\{2, 2\}$	−8·740740$\overline{740}$ (−2)	$B_9\{5, 4\}$	−5·530815271 (0)
$B_5\{5\}$	1·319506173 (0)	$B_9\{5, 2, 2\}$	1·694740776 (0)
$B_5\{3, 2\}$	−2·291358025 (−1)	$B_9\{4, 3, 2\}$	1·761424967 (0)
$B_6\{6\}$	3·081805773 (0)	$B_9\{3, 3, 3\}$	1·383532334 (−1)
$B_6\{4, 2\}$	−6·375839422 (−1)	$B_9\{3, 2, 2, 2\}$	−1·843480227 (−1)
$B_6\{3, 3\}$	−1·478408779 (−1)	$B_{10}\{10\}$	1·467127366 (2)
$B_6\{2, 2, 2\}$	4·396920579 (−2)	$B_{10}\{8, 2\}$	−3·651916018 (1)
$B_7\{7\}$	7·572679376 (0)	$B_{10}\{7, 3\}$	−1·566748493 (1)
$B_7\{5, 2\}$	−1·580881162 (0)	$B_{10}\{6, 4\}$	−1·514197921 (1)
$B_7\{4, 3\}$	−8·184460681 (−1)	$B_{10}\{6, 2, 2\}$	5·136692327 (0)
$B_7\{3, 2, 2\}$	1·993476106 (−1)	$B_{10}\{5, 5\}$	−6·751609252 (0)
$B_8\{8\}$	1·962135434 (1)	$B_{10}\{5, 3, 2\}$	4·778567395 (0)
$B_8\{6, 2\}$	−4·360514192 (0)	$B_{10}\{4, 4, 2\}$	2·632672094 (0)
$B_8\{5, 3\}$	−2·013883867 (0)	$B_{10}\{4, 3, 3\}$	1·238664169 (0)
$B_8\{4, 4\}$	−1·127717514 (0)	$B_{10}\{4, 2, 2, 2\}$	−6·016064197 (−1)
$B_8\{4, 2, 2\}$	6·130529978 (−1)	$B_{10}\{3, 3, 2, 2\}$	−4·349393303 (−1)
$B_8\{3, 3, 2\}$	2·912761755 (−1)	$B_{10}\{2, 2, 2, 2, 2\}$	2·070637041 (−2)
$B_8\{2, 2, 2, 2\}$	−2·823633530 (−2)		

Table 2.

$B_l\{k_i\}$	BCC	SC
$B_2\{2\}$	$\frac{2}{9}$	$\frac{1}{6}$
$B_4\{4\}$	2·562962$\overline{962}$ (−1)	9·592592$\overline{592}$ (−2)
$B_4\{2, 2\}$	−3·851851$\overline{851}$ (−2)	−2·148148$\overline{148}$ (−2)
$B_6\{6\}$	5·378931161 (−1)	1·044035721 (−1)
$B_6\{4, 2\}$	−1·628353070 (−1)	−4·363147728 (−2)
$B_6\{2, 2, 2\}$	1·273275291 (−2)	5·238375185 (−3)
$B_8\{8\}$	1·486407727 (0)	1·526224518 (−1)

Table 2.—(continued)

$B_i\{k_i\}$	BCC	SC
$B_8\{6, 2\}$	−4·968194171 (−1)	−6·910300010 (−2)
$B_8\{4, 4\}$	−1·669375288 (−1)	−2·146338568 (−2)
$B_8\{4, 2, 2\}$	1·026532814 (−1)	1·984829839 (−2)
$B_8\{2, 2, 2, 2\}$	−5·359844971 (−3)	−1·618622099 (−3)
$B_{10}\{10\}$	4·818536169 (0)	2·630251496 (−1)
$B_{10}\{8, 2\}$	−1·806028785 (0)	−1·331792208 (−1)
$B_{10}\{6, 4\}$	−9·999805315 (−1)	−6·670382897 (−2)
$B_{10}\{6, 2, 2\}$	3·826003630 (−1)	3·839992690 (−2)
$B_{10}\{4, 4, 2\}$	2·551331016 (−1)	2·336734209 (−2)
$B_{10}\{4, 2, 2, 2\}$	−6·599351467 (−2)	−9·238279823 (−3)
$B_{10}\{2, 2, 2, 2, 2\}$	2·573781668 (−3)	5·687735907 (−4)

References

Domb C 1960 *Adv. Phys.* **9** 330

Domb C and Green M S 1974 (eds) *Phase Transitions and Critical Phenomena* vol 3 (New York: Academic Press)

Essam J W and Fisher M E 1970 *Rev. Mod. Phys.* **42** 271

Ferrer M, Moore M A and Wortis M 1973 *Phys. Rev.* B **8** 5205

Gerber P R and Fisher M E 1974 *Phys. Rev.* B **10** 4697

—— 1975 submitted to *J. Chem. Phys.*

Jasnow D and Fisher M E 1975 *Phys. Rev.* B submitted for publication

Pfeuty P, Jasnow D and Fisher M E 1974 *Phys. Rev.* B **10** 2088

Stanley H E 1974 *Phase Transitions and Critical Phenomena* vol 3, eds C Domb and M S Green (New York: Academic Press) p 485

Stanley H E and Kaplan T A 1966 *Phys. Rev. Lett.* **16** 981